The integral property of the spheroidal wave functions *

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Abstract

The perturbation method in supersymmetric quantum mechanics (SUSYQM) is used to study whether the spheroidal equations have the shape-invariance property. Expanding the super-potential term by term in the parameter α and solving it, we find that the superpotential loses its shape-invariance property upon to the second term. This first means that we could not solve the spheroidal problems by the SUSQM; further it is not unreasonable to say they are non-solvable in some sense.

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It is well-known that the spherical harmonics are indispensable in many physical process like light scattering, nuclear modeling, signal processing, electromagnetic wave propagation, black hole perturbation theory in four and higher dimensions. The spheroidal harmonics are the extensions of the spherical harmonics in the spheroidal coordinate systems, which are more used in theoretical and technological science. Therefore, the spheroidal harmonics play a premier role in mathematical physics. Nowadays, they have made strong contributions to extensively theoretical and practical applications in science and engineering[1]-[5].

The spheroidal wave equations, as extension of the ordinary spherical harmonics equations are

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+b^2\cos^2\theta-\frac{m^2}{\sin^2\theta}\right]\Theta=-\lambda\Theta \tag{1}$$

Under the boundary condition that Θ is finite at $\theta=0,\ \pi$. This is a kind of Sturm-Liouville boundary problem. Its solution, the eigenfunction with the eigenvalue λ_n, is called the spheroidal wave function [1]-[3].

Usually, by transformation $x=\cos\theta$, one gets the more familiar form of the spheroidal equation

$$\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]+\lambda+b^2x^2-\frac{m^2}{1-x^2}\right]\Theta=0. \tag{2}$$

The equation (1) has two parameters: m, $\alpha=b^2$.

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(1)When $\alpha = 0$, the spheroidal wave functions reduces to the spherical functions (the associated Legendre-functions $P_l^m(x)$). The spherical functions have many well-known simple properties, for example, (1) the Legendre-functions $P_l(x)$ are polynomials; (2) Legendre-functions P_l could be deduced from the first or ground function P_0 from the recursion relation; (3) all the associated Legendre-functions $P_l^m(x)$can be derived from the Legendre-functions $P_l(x)$ by

$$P_l^m(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m P_l(x)}{dx^m}. \tag{3}$$

(2)When $\alpha \neq 0$ and real, the spin-weighted spheroidal wave functions are spheroidal function (prolate or oblate spheroidal function)[1]. They are very different from the spherical functions [1] . When the parameter α is real or pure imaginary, they correspond to the oblate and prolate spheroidal harmonics respectively. Further, the parameter α could be any complex number too. The spherical harmonics have many good property as mentioned above; on the other hand, even the prolate or oblate spheroidal functions have no any of these simple properties or these relations for them have not been found.

In this paper, we only consider the solutions of eq.(1) under the parameters $m = 0$.

Basically, the methods of evaluating the eigenvalues and eigenfunctions of spheroidal harmonics mainly rely on the three-term recurrence relation: one could solve the transcendental equation in continued fraction form or its equivalent or by power series expansion etc[1]-[3],[5]-[8]. For the details of these methods and their advantage and disadvantage, one could see the reference [8]. In the former work, using the perturbation method in supersymmetric quantum mechanics (SUSYQM), we found they have the properties of the shape-invariance when their super-potential expanded in the first term in the parameter α. Just like in the associated-Lendgre functions' case, the shape-invariance property makes it easy to calculate the excited state eigenfunction from the ground one (these obtained eigenfunctions all mean the corresponding ones in the first term of the parameter α) .

Because the shape-invariance property is the integrable property, we could check whether or not the spheroidal equations have the shape-invariance property by the perturbation methods. In the following, we will find that the higher order term of the potentials have no such a good property.

First we give the brief introduction of the supersymmetric quantum mechanics[9], [10].

Supersymmetry offered a possible way of the unified theories, but here attention is mainly paid to supersymmetric quantum mechanics, esp in the solvable potential problems where it gives insight into the factorization method of Infeld and Hull [11]. See reference [9] for review on its development.

In supersymmetric quantum mechanics [9], the Schrödinger equation is

$$H^- \psi^- = -\frac{d^2 \psi^-}{dx^2} + V^-(x)\psi^-(x) = E_- \psi^- \tag{4}$$

with $\hbar = 2m = 1$. We consider the case of unbroken supersymmetry, the ground state is nodeless with zero energy, and suppose the superpotential $W(x)$ is continuous and differentiable, which satisfies the equation

$$V^-(x) = W^2(x) - W'(x), \tag{5}$$

and define

$$\mathcal{A} = \frac{d}{dx} + W(x) \ , \ \mathcal{A}^\dagger = -\frac{d}{dx} + W(x). \tag{6}$$

The corresponding Hamiltonian H^- have a factorized form

$$H^- = \mathcal{A}^\dagger \mathcal{A} \ . \tag{7}$$

The partner potential $V^+(x)$ of V^- is related to the superpotential $W(x)$ by

$$V^+(x) = W^2(x) + W'(x), \tag{8}$$

with the corresponding Hamiltonian H^+

$$H^+\psi^+ = -\frac{d^2\psi^+}{dx^2} + V^-(x)\psi^+(x) = E_+\psi^+ \tag{9}$$

and have a factorized form

$$H^+ = \mathcal{A}\mathcal{A}^\dagger \tag{10}$$

The Hamiltonians H^+ and H^- have exactly the same eigenvalues except that H^- has an additional zero energy eigenstate, that is,

$$E_0^{(-)} = 0 \ , \quad E_{n-1}^{(+)} \ = \ E_n^{(-)}, \quad \psi_{n-1}^{(+)} \propto \mathcal{A}\,\psi_n^{(-)} \quad , \quad \mathcal{A}^\dagger\,\psi_n^{(+)} \propto \psi_{n+1}^{(-)} \quad , \quad n = 1, 2, \ldots . \tag{11}$$

The pair of SUSY partner potentials $V^\pm(x)$ are called shape invariant if they are similar in shape and differ only in the parameters, that is

$$V^+(x; a_1) = V^-(x; a_2) + R(a_1), \tag{12}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of x. One can use the property of shape invariance to obtain the analytic determination of energy eigenvalues and eigenfunctions [11], [10]. Thus for an unbroken supersymmetry, the eigenstates of the potential $V^-(x)$ are:

$$E_0^- = 0, \quad E_n^- = \sum_{k=1}^{n} R(a_k) \tag{13}$$

$$\Psi_0 \propto \exp\left[-\int_{x_0}^{x} W(y, a_1)dy\right] \tag{14}$$

$$\Psi_n^- = \mathcal{A}^\dagger\,(x, a_1)\Psi_{n-1}^-(x, a_2), \quad n = 1, 2, 3, \ldots \tag{15}$$

The shape-invariance property is the integrable property.

In the former work, we have used the perturbation method in supersymmetry quantum to resolve the spheroidal harmonics. In supersymmetry quantum, the central concept is the super-potential W. We had expanded it in the series of the parameter α; and gotten that the super-potential in first order has the shape-invariance property [12].

Because the shape-invariance property is the integrable property, we could check whether or not the spheroidal equations have the shape-invariance property by the perturbation methods. Actually, after expanding the super-potential by the Taylor series of the parameter α, we check its super-invariance property term by term. This is just what we will do in the following.

Now, We briefly introduce our former work [12]. From eq.(1) and by the transformation

$$\Theta = \frac{\Psi}{\sin^{\frac{1}{2}}\theta} \tag{16}$$

we could get

$$\frac{d^2\Psi}{d\theta^2}+\left[\frac{1}{4}+\alpha\cos^2\theta-\frac{m^2-\frac{1}{4}}{\sin^2\theta}+A_s\right]\Psi=0 \tag{17}$$

and the boundary conditions become

$$\Psi|_{\theta=0}=\Psi|_{\theta=\pi}=0 \tag{18}$$

From the equation (17), we know the potential is

$$V(\theta,\alpha,m)=-\frac{1}{4}-\alpha\cos^2\theta+\frac{m^2-\frac{1}{4}}{\sin^2\theta} \tag{19}$$

When the absolute value of α is small, the super-potential W could be expanded as series of the parameter α, that is,

$$W=W_0+\alpha W_1+\alpha^2 W_2+\alpha^3 W_3+\dots. \tag{20}$$

$$\begin{aligned} W^2 &- W'=W_0^2-W_0'+\alpha\left(2W_0W_1-W_1'\right)+\alpha^2\left(2W_0W_2+W_1^2-W_2'\right)\\ &+ \alpha^3\left(2W_0W_3+2W_1W_2-W_3'\right)+\alpha^4\left(2W_0W_4+2W_1W_3+W_2^2-W_4'\right)+\dots. \end{aligned} \tag{21}$$

We can write the perturbation equation as

$$W^2-W'=V(\theta,\alpha,m)-\sum_{n=0}^{\infty}2E_{0n}\alpha^n=-\frac{1}{4}+\frac{m^2-\frac{1}{4}}{\sin^2\theta}-\alpha\cos^2\theta-\sum_{n=0}^{\infty}2E_{0n}\alpha^n \tag{22}$$

there two lower index in the parameter E_{0n}, the index 0 means belongs to the ground state, the other index n means the nth term in parameter α. The last term $\sum_{n=0}^{\infty}2E_{on}\alpha^n$ is subtracted from the above equation in order to make the ground state energy actually zero for the application of the theory of SUSYQM. Later, we have to add the term to our calculated eigen-energy. Compare the equations (21), (19), and (22), one could get

$$W_0^2-W_0'=-\frac{1}{4}+\frac{m^2-\frac{1}{4}}{\sin^2\theta}-2E_{00} \tag{23}$$

$$2W_0W_1-W_1'=-\alpha\cos^2\theta-2E_{01} \tag{24}$$

$$2W_0W_2+W_1^2-W_2'=-2E_{02} \tag{25}$$

$$2W_0W_3+2W_1W_2-W_3'=-2E_{03} \tag{26}$$

$$2W_0W_4+2W_1W_3+W_2^2-W_4'=-2E_{04} \tag{27}$$

$$\vdots \tag{28}$$

From the eq.(23), we get

$$W_0=-\frac{1}{2}\cot\theta,\ 2E_{00}=0. \tag{29}$$

Then, we can write the other equations more concisely

$$W_1' + \cot\theta W_1 = \cos^2\theta + 2E_{01} \tag{30}$$
$$W_2' + \cot\theta W_2 = W_1^2 + 2E_{02} \tag{31}$$
$$W_3' + \cot\theta W_3 = 2W_1W_2 + 2E_{03} \tag{32}$$
$$W_4' + \cot\theta W_4 = 2W_1W_3 + W_2^2 + 2E_{04} \tag{33}$$
$$W_5' + \cot\theta W_5 = 2W_1W_4 + 2W_2W_3 + 2E_{05} \tag{34}$$
$$\vdots \tag{35}$$

The former work indicated that

$$W_1 = \frac{1}{3}\sin\theta\cos\theta. \tag{36}$$

Similarly, it is easy to get

$$W_2 = \frac{\bar{A}_2}{\sin\theta} \tag{37}$$

with

$$\bar{A}_2 = \frac{1}{45}\cos^5\theta - \frac{1}{27}\cos^3\theta - 2E_{02}\cos\theta \tag{38}$$

with the eq.(14), the ground state function or eigenfunction in the second order is

$$\begin{aligned}\Psi_0 &= (sin\theta)^{\frac{1}{2}}\exp\left(-\frac{\alpha\sin\theta}{6}\right)\exp\left[-\alpha^2\int^\theta W_2 d\theta + O(\alpha^3)\right] \\ &= (sin\theta)^{\frac{1}{2}+2E_{02}+\frac{2}{135}}\exp\left(-\frac{\alpha\sin^2\theta}{6}\right) \\ &* \exp\left[\alpha^2\left(\frac{1}{270}\sin^2\theta - \frac{1}{180}\sin^4\theta\right) + O(\alpha^3)\right].\end{aligned} \tag{39}$$

By the eq.(16), the ground spheroidal harmonics S_0 upon to the first order is

$$\Theta_0 = (sin\theta)^{\alpha^2(\frac{2}{135}+2E_{02})}\exp\left(-\frac{\alpha\sin^2\theta}{6}\right) * \exp\left[\alpha^2\left(\frac{1}{270}\sin^2\theta - \frac{1}{180}\sin^4\theta\right) + O(\alpha^3)\right]. \tag{40}$$

With the aid of $x = \cos\theta$, it could be written as

$$\Theta_0 = \left(1-x^2\right)^{\frac{\alpha^2}{2}(\frac{2}{135}+2E_{02})}\exp\left(-\frac{\alpha(1-x^2)}{6}\right) * \exp\left[\alpha^2\left(\frac{(1-x^2)}{270} - \frac{(1-x^2)^2}{180}\right) + O(\alpha^3)\right]. \tag{41}$$

Going back to eq.(ref3), the power number $\frac{2}{135} + 2E_{02}$ must not be less than zero by the boundary condition. Furthermore, if it be greater than zero, we would get $\Theta_0|_{x=\pm1} = 0$; then we also obtain $\frac{d\Theta_0}{dx}|_{x=\pm1} = 0$ by eq.(2). Because the eigenfunction of the zero order term (or the un-perturbation eigenfunction) is not zero at $x = \pm1$, and the ground eigenfunction is the analytic function of the parameter α, it could not be zero at $x = \pm1$. This in turn induce $\frac{2}{135} + 2E_{02} = 0$, that is

$$E_{02} = -\frac{1}{135} \tag{42}$$

.

Therefore,

$$\Psi_0 = (sin\theta)^{\frac{1}{2}} \exp\left[-\frac{\alpha \sin^2\theta}{6} + \alpha^2\left(\frac{1}{270}\sin^2\theta - \frac{1}{180}\sin^4\theta\right) + O(\alpha^3)\right] \tag{43}$$

By eq.(42), we get

$$W_2 = -\frac{1}{135}\sin\theta\cos\theta + \frac{1}{45}\cos\theta\sin^3\theta \tag{44}$$

Because the good result of the quantity W_1 in the form of the equation (36), the potential upon to the first order have the property of the shape-invariance [12]. In order to check whether this property is sustained upon to the second order, we rewrite the super-potential W as

$$W = A_1W_0 + \alpha B_1W_1 + \alpha^2C_1W_2 + O(\alpha^3) \tag{45}$$

then

$$\begin{aligned} V_-(A_1, B_1,\ C_1) &= W^2 - W' \\ &= V_0^-(A_1) + \alpha V_1^-(A_1, B_1) + \alpha^2 V_2^-(A_1, B_1,\ C_1) + O(\alpha^3) \end{aligned} \tag{46}$$

with

$$V_0^-(A_1) = (\frac{A_1^2}{4} - \frac{A_1}{2})\csc^2\theta - \frac{A_1^2}{4} \tag{47}$$

$$V_1^-(A_1, B_1) = -\frac{B_1}{3}(A_1+2)\cos^2\theta + \frac{B_1}{3} \tag{48}$$

$$V_2^-(A_1, B_1, C_1) = \frac{C_1(A_1-1)}{135}\cos^2\theta + \left(\frac{B_1^2}{9} - \frac{C_1(A_1+4)}{45}\right)\sin^2\theta\cos^2\theta + \frac{2C_1}{135} \tag{49}$$

and

$$\begin{aligned} V^+(A_1, B_1, C_1) &= W^2 + W' \\ &= V_0^+(A_1) + \alpha V_1^+(A_1, B_1) + \alpha^2 V_2^+(A_1, B_1,\ C_1) + O(\alpha^3) \end{aligned} \tag{50}$$

with

$$V_0^+(A_1) = (\frac{A_1^2}{4} + \frac{A_1}{2})\csc^2\theta - \frac{A_1^2}{4} \tag{51}$$

$$V_1^+(A_1, B_1) = -\frac{B_1}{3}(A_1-2)\cos^2\theta - \frac{B_1}{3} \tag{52}$$

$$V_2^+(A_1, B_1, C_1) = \frac{C_1(A_1+1)}{135}\cos^2\theta + \left(\frac{B_1^2}{9} - \frac{C_1(A_1-4)}{45}\right)\sin^2\theta\cos^2\theta - \frac{2C_1}{135} \tag{53}$$

The super-invariance property demands the following condition must be met,

$$V^+(A_1, B_1, C_1) = V^-(A_2, B_2, C_2) + R(A_1, B_1, C_1), \tag{54}$$

that is

$$\begin{aligned} V_0^+(A_1) &= V_0^-(A_2) + R_0(A_1) && (55) \\ V_1^+(A_1, B_1) &= V_1^-(A_2, B_2) + R_1(A_1, B_1) && (56) \\ V_2^+(A_1, B_1,\ C_1) &= V_2^-(A_2, B_2, C_2) + R_2(A_1, B_1, C_1) && (57) \end{aligned}$$

and

$$R(A_1, B_1, C_1) = R_0(A_1) + \alpha R_1(A_1, B_1) + \alpha^2 R_2(A_1, B_1, C_1) + O(\alpha^3) \tag{58}$$

In reference [12], the conditions of eqs (55), (56) could be satisfied with the parameters $A_2,\ B_2$ defined as

$$A_2 = f_1(A_1) = A_1 + 2, \quad B_2 = f_2(A_1, B_1) = \frac{A_1 - 2}{A_1 + 4} B_1 \tag{59}$$

Now it is easy to see that the eq.(57) can not be met under the relations of eq.(59) with $A_1 = B_1 = C_1 = 1$.

In conclusion, This in turn shows that the shape-invariance relation could not be attained under the second term of perturbations. What is the real meaning of the above conclusion? At least we can say that the spheroidal equation has no shape-invariance property with respect to the series expansion of the parameter α. Perhaps it is not unreasonable to say that it is non-solvable in some sense.

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